Elasticity-driven interaction between vortices in type-II superconductors

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The contribution to the vortex lattice energy which is due to the vortex-induced strains is calculated, covering all the magnetic-field range which defines the vortex state. The comparison with previously reported results shows that, in most of the vortex state, it has been notably underestimated until now. The assumption that only the vortex cores induce strains leads to this underestimation. In fact, all spatial variations of the order parameter induce strain. Core regions are important because here the order parameter varies strongly, but the non-core regions (smooth variations) might be even more important if their extension is large enough. It proves that in high-κ superconductors, in which the supercurrent regions with smooth variation of the order parameter are much more extended than the cores, the major contribution to the vortex-induced strains is due to the non-core regions.

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I. INTRODUCTION

Since a long time ago, much attention has been paid to the role of long-range strain fields in the vortex state of type-II superconductors. It is well known, for instance, that interaction between defect-induced strains and vortices causes pinning phenomena. These phenomena have been extensively studied almost since Abrikosov predicted the superconducting vortices (see, e.g., Refs. 1–8). It is also known that vortex-induced strains give a contribution to the energies of the vortex lattices (VL's). This contribution proves to be essential when discussing the observed correlations[9] between VL's and crystal lattices in anisotropic superconductors.[10–12] The vortex-induced strains might also be important in vortex inertia, because they contribute to the effective masses of vortices.[13]

In this paper, we calculate the contribution to the VL energy due to the vortex-induced strains. Comparison with the previously reported calculations[10–12] shows that, for magnetic fields not so close to the upper critical field H_{c2}, this contribution has been notably underestimated until now. The reason of such underestimation is connected with the fact that, contrary to what is assumed in many occasions, the vortex core is not the primary source of strain when the Ginzburg-Landau parameter κ of the superconductor is large.

To clarify this point we shall revise, first of all, the strain induced by a single vortex. This strain is due to all the spatial variations of the density of superconducting electrons that the vortex provokes. The vortex core is a region of strong variations, but is not the only one. There also exists a region of smooth variation which is associated with the presence of superconducting currents. In high-κ superconductors, the size of the latter region is much larger than the core one. Because of this greater extension, the non-core variation in the density of superconducting electrons finally emerges as the main source of strains.

In previous papers,[10–12] the elasticity-driven interaction between vortices was invoked in order to explain observed correlations between VL's and crystal lattices. In fact, it was found strong enough to explain these correlations in $NbSe_2$.[11,12] The proper inclusion of all strain sources increases significantly the strength of this interaction. So all those cases in which previous calculations indicated that the elasticity-driven interaction between vortices was not strong enough should be reconsidered.

Let us mention that we evaluate this interaction taking into account all the elastic degrees of freedom of free samples of finite size, i.e., taking into account both homogeneous and inhomogeneous deformations. In a general case, the elasticity-driven interaction between vortices includes contributions due to both types of deformations. In the elastically isotropic case the contribution due to the inhomogeneous deformations vanishes (they are pure shear deformations). In the anisotropic case, the order of magnitude of the total interaction coincides with that of the contribution due to homogeneous deformations.

The consideration of homogeneous deformations provides us, in addition, a useful technical trick. Its first step is to evaluate the VL energy for elastically isotropic superconductors with infinite shear modulus μ. In this case, the calculations are free of approximations and are almost trivial. If $\mu=\infty$, the only elastic degree of freedom of the sample is its homogeneous dilatation. Therefore, using already known formulas for the VL energy and taking into account the dependence on the dilatation of the corresponding coefficients, the elastic contribution can be easily calculated. As we shall see, any isotropic case can be reproduced from this $\mu=\infty$ one. Moreover, the previously reported results can be easily checked by evaluating them for $\mu=\infty$ and comparing them with those obtained considering this case from the beginning.

Let us mention also that we use the Fourier method when calculating the VL energy in the elastically anisotropic case. This method permits us to satisfy quite easily the (elastic) boundary conditions for a free sample. Thus one avoids to reproduce spurious effects that a lack of attention to these conditions might give. One such effect is, for instance, the sample form dependence of the elasticity-driven interaction between vortices (the same group of authors reported this dependence in Ref. 11 but not in Ref. 12).

The paper is organized as follows. In Sec. II we outline how to account for the elastic effects in the London limit. This question is not so trivial because a wrong interpretation of the London approximation seems to be a reason of the oversight of the importance of the non-core contributions to the elasticity-driven interaction between vortices. In Sec. III we clarify the role that core and non-core regions play in this interaction, discussing in detail the strain field induced by one single vortex. Here we show explicitly that a large value of κ implies that most of the vortex-induced strain is due to the non-core region. In Secs. IV and V we calculate the VL energy taking into account the elasticity-driven interaction between vortices, and compare our results with previously reported ones. In Sec. IV we deal with elastically isotropic superconductors, while elastically anisotropic ones are considered in Sec. V. Finally, in Sec. VI, we discuss possible applications of our results.

II. ON THE ELASTIC EFFECTS WITHIN THE LONDON LIMIT

When studying the influence of the elasticity on the vortex properties, many authors use an assumption which might seem quite natural (see, e.g., Refs. 2,5, and 10–12). It consists of using the "London approximation" introduced by Abrikosov in Ref. 14 (see also Ref. 15). However, the essence of this approximation could easily be misinterpreted. As it is frequently commented, within the London approximation the order-parameter modulus varies significantly only inside of the vortex cores. Since the spontaneous deformation associated with the superconductivity is proportional to square of the order-parameter modulus, it seems natural that only the core regions ($\rho \lesssim \xi$) are essential sources of stresses. It is just what is assumed in Refs. 2,5, and 10–12. However, one has to bear in mind that supercurrents also produce an elastic effect because they diminish the value of the order-parameter modulus. Locally this reduction is small. But since the supercurrents can occupy a very broad region ($\rho \lesssim \lambda_L$), their effect might be comparable and even more important, as it virtually proves to be, than the one of the cores.

To make this point more clear, let us recall how the vortex self-energy per unit length ε_0 is calculated within the London limit.[14,15] Within this limit one assumes that, when calculating the supervelocity v_s from the Ginzburg-Landau equations, the density of superconducting electrons (the square of the order parameter modulus f^2) is constant in the corresponding equation. This makes it possible to find explicitly the spatial distribution of the supervelocity. After doing so, one can proceed in two different ways:

(i) Following de Gennes,[16] the vortex self-energy is presented as a sum of the magnetic-field energy and the kinetic energy of the superconducting electrons:

$$\varepsilon_0 = \int (H^2 + f^2 v_s^2) d^2\boldsymbol{\rho} \tag{1}$$

[we use here the reduced units, see Refs. 14 and 15, which are analogous of those defined in Eqs. (10) (see below)]. Integration is carried out taking into account the already found supervelocity and considering that the density of superconducting electrons is constant. This approximation is justified by virtue of the high value of κ: f^2 diminishes significantly only at $\rho \lesssim \xi$, whereas v_s^2 does at $\rho \gtrsim \lambda_L$.

(ii) Following Abrikosov,[14,15] the vortex self-energy is calculated from the exact formula

$$\varepsilon_0 = \int \left[H^2 + \frac{1}{2}(1 - f^4) \right] d^2\boldsymbol{\rho} \tag{2}$$

(as before, we use here dimensionless quantities). The principal part of this integral arises from the second term and it is associated with distances much larger than ξ. In other words, those variations of f that takes place out of the vortex core are now essential.

As we see, to assume that within the London approximation f is constant out of the vortex cores is not always correct. But, as we have pointed out, this is just the assumption that unfortunately many authors made. For example, when studying the interaction between vortices and lattice defects, Miyahara *et al.*[5] considered integrals which are similar to Eq. (2) but, at the same time, neglected all the spatial variations of f at $\rho \gtrsim \xi$.

It is quite surprising that this assumption has not been critically revised up to now, especially by noting that, in principle, the importance of the out-of-core region for the elastic effects could be understood long ago. Galaiko[4] considered the interaction between vortices and dislocation-induced strains. He found that this interaction depends not only on ξ, but also on λ_L. However, he did not comment on Ref. 2 and discussed neither the vortex-induced strain nor the strain-induced interaction between vortices. Reference 8 is a recent example in which the out-of-core region is taken into account when studying an elasticity related problem: the structure of a superconducting vortex pinned by a screw dislocation.

III. ONE SINGLE VORTEX

A. Vortex-induced strain

Let us proceed with the calculation of the strain field induced by one single vortex. When doing so, we shall account for all the spatial variations, core and non-core ones, that are associated with the vortex.

The free energy can be presented as

$$F = F_1 + F_2 = \frac{1}{v} \int (\mathcal{F}_1 + \mathcal{F}_2) dv, \tag{3}$$

where v is the volume of the system, and

$$\mathcal{F}_1 = \frac{H^2}{8\pi} + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{4m}\left| \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right)\Psi \right|^2, \tag{4a}$$

$$\mathcal{F}_2 = \alpha_{ij}|\Psi|^2 u_{ij} + \frac{1}{2}\lambda_{ijkl} u_{ij} u_{kl}. \tag{4b}$$

Here and below, summation over double indices is implied.

The equations of equilibrium read[15,17]

$$\left[a+b\left|\Psi\right|^2+\alpha_{ij}u_{ij}+\frac{1}{4m}\left(-i\hbar\nabla-\frac{2e}{c}\mathbf{A}\right)^2\right]\Psi=0, \tag{5a}$$

$$\nabla\times\mathbf{H}=\frac{4\pi e}{mc}\left[\frac{\hbar}{2i}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)-\frac{2e}{c}\left|\Psi\right|^2\mathbf{A}\right], \tag{5b}$$

$$\lambda_{ijkl}\langle u_{kl}\rangle+\alpha_{ij}\langle|\Psi|^2\rangle=0, \tag{5c}$$

$$\frac{\partial}{\partial x_j}(\lambda_{ijkl}u_{kl}+\alpha_{ij}|\Psi|^2)=0, \tag{5d}$$

where $\langle\cdots\rangle$ means volume average. We shall look for the solution of these equations for the case of a single vortex. The z axis of the coordinate frame we choose is parallel to the vortex. The crystal frame is obtained from this coordinate frame by rotation.

It is clear that far enough from the vortex both the order parameter and the strain tensor tend to constant values; say Ψ_s and u_{ij}^s, respectively. Assuming that $\langle|\Psi|^2\rangle\simeq|\Psi_s|^2$, the equations of equilibrium reduce to

$$a+b|\Psi_s|^2+\alpha_{ij}u_{ij}^s=0, \tag{6a}$$

$$\lambda_{ijkl}u_{kl}^s+\alpha_{ij}|\Psi_s|^2=0. \tag{6b}$$

In consequence,

$$|\Psi_s|^2=-a/b^*, \tag{7}$$

$$u_{ij}^s=a\alpha_{kl}\lambda_{ijkl}^{-1}/b^*, \tag{8}$$

where $b^*=b-\alpha_{ij}\alpha_{kl}\lambda_{ijkl}^{-1}$ (λ_{ijkl}^{-1} is given by $\lambda_{ijkl}^{-1}\lambda_{ijk'l'}=\delta_{kk'}\delta_{ll'}$). These values correspond to those that one obtains in the homogeneous superconducting phase without magnetic field.

To evaluate the effects associated with the vortex let us put $u_{ij}=u_{ij}^s+u_{ij}^v$. Thus we can rewrite the equation of equilibrium (5a) as

$$\left[1-\frac{|\Psi|^2}{|\Psi_s|^2}-\frac{\alpha_{ij}u_{ij}^v}{|\Psi_s|^2b}+\xi^2\left(\nabla-\frac{2ie}{\hbar c}\mathbf{A}\right)^2\right]\Psi=0, \tag{9}$$

where $\xi^2=\hbar^2/(4m|\Psi_s|^2b)$. It is convenient to introduce the following notation:

$$\lambda_L=\sqrt{\frac{mc^2}{8\pi e|\Psi_s|^2}},\quad H_c=\frac{\hbar c}{2\sqrt{2}e\xi\lambda_L},$$

$$\Psi'=\frac{\Psi}{\Psi_s},\quad \mathbf{r}'=\frac{\mathbf{r}}{\lambda_L}, \tag{10}$$

$$\mathbf{H}'=\frac{\mathbf{H}}{\sqrt{2}H_c},\quad \mathbf{A}'=\frac{\mathbf{A}}{\sqrt{2}H_c\lambda_L},$$

$$\hat{\alpha}'=\frac{\hat{\alpha}}{|\Psi_s|^2b},\quad \hat{\lambda}'=\frac{\hat{\lambda}}{|\Psi_s|^4b}.$$

Thus the equations of equilibrium can be written as (we omit the primes in the following)

$$(1-v_s^2-\alpha_{ij}u_{ij}^v)f-f^3=-\kappa^{-2}\triangle f, \tag{11a}$$

$$\nabla\times\mathbf{H}=\mathbf{v}_sf^2, \tag{11b}$$

$$\lambda_{ijkl}\langle u_{kl}\rangle+\alpha_{ij}\langle f^2\rangle=0, \tag{11c}$$

$$\frac{\partial}{\partial x_j}(\lambda_{ijkl}u_{kl}+\alpha_{ij}f^2)=0, \tag{11d}$$

where the order parameter has been expressed as $\Psi=fe^{i\chi}$, with $\mathbf{v}_s=\kappa^{-1}\nabla\chi-\mathbf{A}$ the above-mentioned supervelocity. Here $\kappa=\lambda_L/\xi$ represents the Ginzburg-Landau parameter in our case, which does not differ substantially from the one without account for the strains ($b^*\simeq b$).

The spatial distribution of the supervelocity $\mathbf{v}_s$ can be obtained from Eq. (11b) by assuming that f is constant there, i.e., within the London limit. Thus one finds that $v_s=\kappa^{-1}K_1(\rho)$, where K_1 is the MacDonald function (see, e.g., Ref. 15).

In Eq. (11a), the term with u_{ij}^v results to be of order $\hat{\alpha}^2$ because of Eqs. (11c) and (11d). Since $\hat{\alpha}$ is small,[18] the vortex-induced strain can be calculated to the lowest order in $\hat{\alpha}$ neglecting the changes in f induced by the term u_{ij}^v in Eq. (11a). In other words, f^2 in Eqs. (11c) and (11d) can be taken as the solution of Eq. (11a) with $\hat{\alpha}=0$. This solution can be written as $f^2=1-h$, where h represents the vortex contribution. Using the same approximation as that in Ref. 15 we have

$$h(\boldsymbol{\rho})=\begin{cases}v_s^2(\rho) & \rho\gg\kappa^{-1},\\ 1-C(\kappa\rho)^2, & \rho\ll\kappa^{-1},\end{cases} \tag{12}$$

where C is a constant of order unity.

We present the vortex induced strain as[19]

$$u_{ij}^v=\epsilon_{ij}+\frac{i}{2}\sum_{\mathbf{q}\neq 0}[q_iu_j(\mathbf{q})+q_ju_i(\mathbf{q})]e^{i\mathbf{q}\cdot\boldsymbol{\rho}}$$

$$\simeq\epsilon_{ij}+\frac{iA}{8\pi^2}\int d^2\mathbf{q}[q_iu_j(\mathbf{q})+q_ju_i(\mathbf{q})]e^{i\mathbf{q}\cdot\boldsymbol{\rho}}, \tag{13}$$

where A is the area of the sample perpendicular to the vortex. Here ϵ_{ij} accounts for the homogeneous strain induced by the vortex, and $u_i(\mathbf{q})$ is the ith component of the displacement vector in Fourier space. Thus Eqs. (11c) and (11d) can be written as

$$\lambda_{ijkl}\epsilon_{kl}-\alpha_{ij}\langle h\rangle=0, \tag{14a}$$

$$G_{ik}^{-1}(\mathbf{q})u_k(\mathbf{q})+iS_i(\mathbf{q})h(\mathbf{q})=0 \tag{14b}$$

where $S_i(\mathbf{q})=\alpha_{ij}q_j$, $G_{ik}^{-1}(\mathbf{q})=\lambda_{ijkl}q_jq_l$, and $h(\mathbf{q})$ is the Fourier transform of the function (12). For the strain field we have

$$\epsilon_{ij}=\alpha_{kl}\lambda_{ijkl}^{-1}\langle h\rangle, \tag{15a}$$

$$u_i(\mathbf{q})=-iS_k(\mathbf{q})G_{ki}(\mathbf{q})h(\mathbf{q}). \tag{15b}$$

When calculating the strain at a fixed distance ρ from the vortex, the inhomogeneous strain is mainly given by the integrand in Eq. (13) with $q\ll\rho^{-1}$. So the main contribution at long distances ($\rho\gg 1$) arises from $q\ll 1$. For these small $\mathbf{q}$'s, the function $h(\mathbf{q})$ can be split into core and non-core contributions:

$$h_{\text{core}}(\mathbf{q})\simeq\frac{1}{A}\int_0^{\kappa^{-1}}\int_0^{2\pi}(\rho-\kappa^2\rho^3)e^{-iq\rho\cos\theta}d\rho d\theta$$
$$=\frac{2\pi}{A}\int_0^{\kappa^{-1}}(\rho-\kappa^2\rho^3)J_0(q\rho)d\rho\underset{q\ll 1}{=}\frac{\pi}{2A\kappa^2}, \tag{16a}$$

$$h_{\text{non-core}}(\mathbf{q})\simeq\frac{1}{A\kappa^2}\int_{\kappa^{-1}}^{1}\int_0^{2\pi}\rho^{-1}e^{-iq\rho\cos\theta}d\rho d\theta$$
$$=\frac{2\pi}{A\kappa^2}\int_{\kappa^{-1}}^{1}\frac{J_0(q\rho)}{\rho}d\rho\underset{q\ll 1}{=}\frac{2\pi}{A\kappa^2}\ln\kappa \tag{16b}$$

(here we have used the asymptotic form of $v_s\approx 1/(\kappa\rho)$ for $\kappa^{-1}\ll\rho\ll 1$, see Ref. 15).

As a result, at $\rho\gg 1$ the strain tensor can be written as

$$u_{ij}^{v}(\boldsymbol{\rho})=\eta\left[\frac{\alpha_{kl}\lambda_{ijkl}^{-1}}{A}+\int\frac{d^2\mathbf{q}}{(2\pi)^2}q_iS_k(\mathbf{q})G_{kj}(\mathbf{q})e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\right]$$
$$=\eta\left[\frac{\alpha_{kl}\lambda_{ijkl}^{-1}}{A}+\frac{1}{\rho^2}\int_0^{2\pi}\Theta_{ij}(\theta_{\mathbf{q}})d\theta_{\mathbf{q}}\right], \tag{17}$$

where $\eta=\int h(\rho)d^2\boldsymbol{\rho}=\pi(1+4\ln\kappa)/(2\kappa^2)$, and Θ_{ij} is a tensor which depends only on the angle $\theta_{\mathbf{q}}$ ($\mathbf{q}\cdot\boldsymbol{\rho}=q\rho\cos\theta_{\mathbf{q}}$). If the sample is large enough the first term in Eq. (17) can be neglected. But we retain it because, when dealing with the strain-induced interaction (see below), its contribution becomes significant (this fact is well known in the theory of point defects, see, e.g., Ref. 20). Note that the non-core contribution to η, i.e., the logarithmic term, could also be obtained from the well-known expression of the vortex self-energy: according to Abrikosov,[15] $\varepsilon_0\simeq\frac{1}{2}\int(1-f^4)d^2\boldsymbol{\rho}\simeq 2\pi\int_{\kappa^{-1}}^{1}h(\rho)\rho d\rho=2\pi\kappa^{-2}\ln\kappa$.

Kogan *et al.*[11] obtained a similar expression for vortex-induced strain considering an infinite medium. In such a case, the first term of Eq. (17) vanishes completely. But the main difference between Eq. (17) and the expression reported by Kogan *et al.*[11] resides in the corresponding values of η. Assuming that only the vortex core induces strain, Kogan *et al.* reported a value π/κ^2. So they overlooked the logarithmic term in $\eta=\pi(1+4\ln\kappa)/(2\kappa^2)$ which arises from the non-core contributions. This implies that in the case of high-κ superconductors, Kogan *et al.* strongly underestimated the vortex-induced strain.

B. Elasticity-driven interaction between vortices: Qualitative estimations

Let us now estimate the interaction energy of a VL which is associated with the vortex-induced strains. As we have pointed out before, the inhomogeneous part of these strains has been calculated previously neglecting non-core contributions (see, e.g., Ref. 11). If the distance between vortices is much longer than λ_L, to take into account these non-core contributions reduces to modify the previously found strains by a factor. In consequence, the interaction energy that one obtains considering both core and non-core contributions coincides, up to the corresponding factor, with previously reported ones. Kogan *et al.*,[11] for instance, evaluated the interaction energy of a VL by summing up all pairwise contributions. Modifying this interaction energy by including the non-core contributions, one can see that

$$F_{\text{int}}^{(\text{nh})}\sim-\frac{(1+4\ln\kappa)^2}{\kappa^2}\frac{\Delta K}{K}B^2. \tag{18}$$

Here $\Delta K/K$ stands for the order of magnitude of the relative change in the elastic moduli due to the normal-superconducting transition, and B represents the magnetic induction.

The interaction between vortices due to homogeneous strains can be easily estimated as follows. It is clear that N vortices will induce a total (homogeneous) strain $N\epsilon$, where ϵ is given by Eq. (15a), if the distance between them is large enough. When substituting this strain in the corresponding terms of VL energy: $-\alpha(N\epsilon)(N\langle h\rangle)+\lambda(N\epsilon)^2/2$, one obtains $-n^2\eta^2\alpha^2/(2\lambda)$, where $n=N/A$ is the vortex density (recall that $\eta=A\langle h\rangle$). This is precisely the interaction term that we are looking for. Taking into account that the vortex density is $n=\kappa B/(2\pi)$, and $\alpha^2/\lambda=\Delta K/K$ [recall that we are using the dimensionless units defined in Eq. (10)]; this interaction can be estimated as

$$F_{\text{int}}^{(\text{h})}\sim-\frac{(1+4\ln\kappa)^2}{\kappa^2}\frac{\Delta K}{K}B^2. \tag{19}$$

As we see, the order of magnitude of both interaction terms Eqs. (18) and (19) coincides. Consequently, either of them gives us an estimate of the order of magnitude of the total interaction energy.

IV. VORTEX LATTICE: ELASTICALLY ISOTROPIC MEDIUM

It is convenient to begin the treatment of VL's considering the case of elastically isotropic superconductors. In this case, the elastic contribution to the VL energy can be obtained, without any new approximation, from already known formulas for the VL energy. Such formulas are available for the regions $H\approx H_{c1}$, $H\approx H_{c2}$ (Refs. 14 and 15) and for intermediate fields $H_{c1}\ll H\ll H_{c2}$ (Refs. 16 and 21). They reasonably match at the boundaries of the corresponding regions (see Appendix). This permits us to study the elastic effects in isotropic superconductors with the same accuracy. We begin with the case $\mu=\infty$ where the calculations are elemental.

A. Infinite shear modulus

The only elastic degree of freedom of a system which shear modulus is infinite is its homogeneous dilatation. If the system is not clamped this homogeneous dilatation, say u, must be understood as a variational parameter. In the free energy (3), this variational parameter modifies the coefficient of the term $|\Psi|^2$, which can be rewritten as $a(u)=a+\alpha u$.

Let us fix the parameter u for awhile, i.e., let us consider for a time a clamped sample. Thus after minimizing with respect to all degrees of freedom but u, the free energy of the VL with respect to that of the superconducting state can be written as a sum of two terms: a u-dependent VL energy via the coefficient $a(u)$, and the elastic energy. That is,

$$F=F_{\mathrm{VL}}(u)+\frac{K}{2}u^2, \tag{20}$$

where F_{VL} is given by (see, e.g., Ref. 22 and the references therein):

$$F_{\mathrm{VL}}\simeq\begin{cases}\dfrac{BH_{c1}}{4\pi} & \text{(I)},\\ \dfrac{1}{8\pi}\left[B^2+BH_{c1}\dfrac{\ln(\nu d/\xi)^2}{\ln\kappa}\right] & \text{(I–II)},\\ \dfrac{1}{8\pi}\left[B^2-\dfrac{(H_{c2}-B)^2}{1+(2\kappa^2-1)\beta_A}\right] & \text{(II)},\end{cases} \tag{21}$$

over the corresponding regions of magnetic fields defined as (I): $H\approx H_{c1}$, (I–II): $H_{c1}\ll H\ll H_{c2}$, and (II): $H\approx H_{c2}$. Here $\beta_A=\langle\Psi^4\rangle/\langle\Psi^2\rangle^2=1.16$ for a triangular VL, and $2\ln\nu=2(\gamma-1)+\ln[\sqrt{3}/(8\pi)]$, where $\gamma(=0.57772\cdots)$ is the Euler's constant. The magnetic induction B and the distance between vortices d are such that $B=2\phi_0/(\sqrt{3}d^2)$ in a triangular VL, where ϕ_0 is the flux quantum. B is given as a function of the magnetic field in the Appendix [see Eq. (A1)].

Recall that in high-κ superconductors one has the following relationships (see, e.g., Refs. 15 and 22):

$$H_{c1}=\frac{\ln\kappa}{2\kappa^2}H_{c2}, \tag{22}$$

$$\left(\frac{d}{\xi}\right)^2=\frac{4\pi}{\sqrt{3}}\frac{H_{c2}}{B}=\frac{8\pi}{\sqrt{3}}\frac{\kappa^2}{\ln\kappa}\frac{H_{c1}}{B}. \tag{23}$$

The critical magnetic fields entering all above expressions are u-dependent magnitudes:

$$H_{c1}(u)=\frac{\ln\kappa}{\sqrt{2}\kappa}H_c(u)=H^{\circ}_{c1}+H'_{c1}u, \tag{24}$$

$$H_{c2}(u)=\sqrt{2}\kappa H_c(u)=H^{\circ}_{c2}+H'_{c2}u, \tag{25}$$

where $H_c(u)=2a(u)\sqrt{\pi/b}=H^{\circ}_c+H'_c u$ (with $H^{\circ}_c=2a\sqrt{\pi/b}$ and $H'_c=2\alpha\sqrt{\pi/b}$). The ratio d/ξ is also a u-dependent magnitude, which can be written as $d/\xi=d/\xi^{\circ}+(d/\xi')u$. But the Ginzburg-Landau parameter κ is independent of u because it does not depend on the coefficient $a(u)$.

Let us now proceed to minimize the free energy (20) with respect to u, i.e., to take into account that the sample is in fact unclamped. After doing so, we obtain

$$F\simeq\begin{cases}\dfrac{1}{8\pi}(2BH^{\circ}_{c1}-\delta_{\mathrm{I}}B^2), & \text{(I)},\\ \dfrac{1}{8\pi}\left[B^2+BH^{\circ}_{c1}\dfrac{\ln(\nu d/\xi^{\circ})^2}{\ln\kappa}-\delta_{\mathrm{I\text{-}II}}B^2\right] & \text{(I–II)},\\ \dfrac{1}{8\pi}\left[B^2-\dfrac{(H^{\circ}_{c2}-B)^2}{1+(2\kappa^2-1)\beta_A-\beta_e}\right] & \text{(II)},\end{cases} \tag{26}$$

where

$$\delta_{\mathrm{I}}=\frac{\ln^2\kappa}{2\kappa^2}\frac{\Delta K}{K}, \tag{27}$$

$$\delta_{\mathrm{I\text{-}II}}=\frac{[1+2\ln(\nu d/\xi^{\circ})]^2}{16\pi K+2\sqrt{2}H_c'^2B/\kappa H^{\circ}_c}\frac{H_c'^2}{\kappa^2}\simeq\frac{\ln^2(d/\xi^{\circ})}{4\kappa^2}\frac{\Delta K}{K}, \tag{28}$$

$$\beta_e=2\kappa^2\frac{\Delta K}{K}. \tag{29}$$

Here it has been taken into account that $H_c'^2/(4\pi K)=\Delta K/K$ is the relative change in the bulk modulus due to the normal-superconducting transition. Because this relative change is usually very small $\Delta K/K\ll 1$, the expression for the region (II) in Eq. (26) can be written as

$$F\simeq\frac{1}{8\pi}\left[B^2-\frac{(H^{\circ}_{c2}-B)^2}{1+(2\kappa^2-1)\beta_A}-\delta_{\mathrm{II}}(H^{\circ}_{c2}-B)^2\right], \tag{30}$$

where

$$\delta_{\mathrm{II}}=\frac{\beta_e}{[1+(2\kappa^2-1)\beta_A]^2}\simeq\frac{1}{2\kappa^2}\frac{\Delta K}{K}. \tag{31}$$

In all above expressions for the free energy, one can identify a term

$$F_{\mathrm{int}}=-\frac{\delta B^2}{8\pi}, \tag{32}$$

which describes an attractive interaction between vortices. Since the different expressions in Eq. (21) match one each other at the boundaries of the corresponding regions (see Appendix), the coefficient δ can be presented as

$$\delta\approx\frac{[\tilde{\nu}+\ln(d/\xi^{\circ})]^2}{2\kappa^2}\frac{\Delta K}{K} \tag{33}$$

taking into account that the ratio d/ξ° must be replaced by κ if $d\gtrsim\lambda_L$, where $\tilde{\nu}$ is a constant of order of unity (see Fig. 1).

According to what we have seen in the preceding section, the logarithmic contribution to the coefficient δ is due to

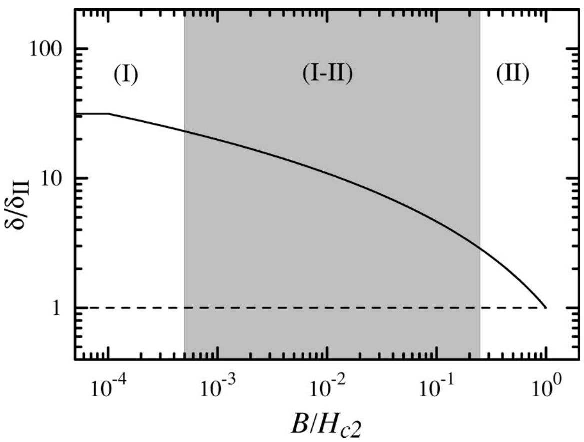

FIG. 1. Log-log plot of the coefficient δ of the attraction term $\sim -\delta B^2$ of the free energy as a function of the magnetic induction, taking into account (solid line) and neglecting (dashed line) non-core contributions. The regions indicated as (I), (I–II), and (II) (see text) correspond to $\kappa \simeq 100$ (note that $H_{c1} \simeq 10^{-4} H_{c2}$ in this case).

non-core effects. As we show in Fig. 1, the neglecting of these non-core effects leads to underestimation the elasticity-driven interaction between vortices. And by virtue of the high value of κ, such a underestimation is quite significant in almost all the mixed state.

B. Finite shear moduli

It is quite straightforward to extend the results that we have obtained for infinite shear modulus $\mu = \infty$, to the most general isotropic case. Note that minimizing the free energy (3) with respect to all elastic degrees of freedom one obtains

$$F_2 = -\frac{\alpha^2}{2K_{4/3}} \langle |\Psi|^4 \rangle - \frac{\alpha^2}{2K} \frac{4\mu}{3K_{4/3}} \langle |\Psi|^2 \rangle^2, \quad (34)$$

where $K_{4/3} = K + 4\mu/3$. The first term of this expression renormalizes the coefficient b of Eq. (1). This renormalization disappears in the limit $\mu \to \infty$. The second term makes that the free energy becomes a nonlocal functional. This nonlocality remains as long as the shear modulus does not vanish.

From this functional one could recover the free energy (26) by ascribing the coefficients in Eq. (34) the values they assume in the case $\mu = \infty$, i.e., $\alpha^2/(2K_{4/3}) = 0$ and $4\mu/(3K_{4/3}) = 1$. This consideration does not change the functional form of Eq. (34) (only the coefficients change). So one concludes that the free energy density of the VL in any isotropic type-II superconductor has the form of Eq. (26) with the corresponding renormalized constants:

$$b \to b - \alpha^2/K_{4/3}, \quad (35)$$

$$(\alpha^2/K) \to (\alpha^2/K)[4\mu/(3K_{4/3})]. \quad (36)$$

Note that, because the resulting coefficient δ in Eq. (32) vanishes if $\mu = 0$, it can be said that the elasticity-driven interaction between vortices is associated with the solid-state elasticity.

C. Comparison with previously reported results

Let us start this section by comparing our results with those reported in Refs. 10 and 11. In these references, intermediate fields far from T_c are considered. Although strictly speaking the Ginzburg-Landau approach that we use is not valid far from T_c, it still gives correctly the orders of magnitude. So the comparison still makes sense. We mention also that in Refs. 10 and 11 the homogeneous part of the strains are omitted. In Ref. 10 this omission is mentioned explicitly, while in Ref. 11 it follows from the fact that they consider infinite samples when calculating the interaction between vortex pairs. Therefore, as we argued in Sec. III B, we can only compare the order of magnitude (see below for a more detailed comparison). Such a comparison reveals that, as a result of neglecting the non-core contributions in the interaction energy, this energy is notably underestimated in Refs. 10 and 11 through most of the mixed state (see Fig. 1). Such an underestimation is at least by a factor $\sim \ln^2\kappa$ close to H_{c1}.

In Ref. 12, treating the case $H \approx H_{c2}$, both homogeneous and inhomogeneous strains are seemingly taken into account. According to Eq. (28) of this reference, the free energy in the isotropic case should be of the form

$$F = \frac{1}{8\pi}\left[B^2 - \frac{1 + (2\kappa^2 - 1)\beta_A - 4\kappa^2\beta_2}{[1 + (2\kappa^2 - 1)\beta_A + 4\kappa^2\beta_2]^2} (H^{\circ}_{c2} - B)^2 \right], \quad (37)$$

where

$$\beta_2 = -\frac{\alpha^2}{(K_{4/3})b}\beta_A. \quad (38)$$

Because of the smallness of β_2, this expression can be approximated to

$$F \simeq \frac{1}{8\pi}\left[B^2 - \frac{(H^{\circ}_{c2} - B)^2}{1 + (2\kappa^2 - 1)\beta_A} - \delta_2 (H^{\circ}_{c2} - B)^2 \right], \quad (39)$$

where $\delta_2 = -12\kappa^2\beta_2/[1 + (2\kappa^2 - 1)\beta_A]^2$.

One can see that the last term of this expression and the last one of Eq. (30) differ in a numerical factor. A deeper inspection of Eq. (39) reveals that it is erroneous: the coefficient δ_2 (i) vanishes if $\mu = \infty$ and (ii) remains finite if $\mu = 0$. Since the strain-driven interaction between vortices is due to the specific features of the solid-state elasticity, the results one obtains from Ref. 12 in two limiting cases cannot be correct. This motivates us to reconsider the problem treated in Ref. 12 (Sec. V B below).

V. VORTEX LATTICE: ELASTICALLY ANISOTROPIC MEDIUM

We now proceed to calculate the VL energy in elastically anisotropic superconductors. First of all, let us reconsider the free energy (3) and, as usual (see Ref. 15), integrate by parts term with $\nabla\Psi$. Thus, after using the Gauss's theorem and the boundary condition

$$\mathbf{n}\cdot\left(-i\hbar\nabla-\frac{2e}{c}\mathbf{A}\right)\Psi|_{\Sigma}=0$$

($\mathbf{n}$ is the unit vector of the normal to the surface Σ), one finds that

$$\frac{1}{4m}\int\left|\left(-i\hbar\nabla-\frac{2e}{c}\mathbf{A}\right)\Psi\right|^2dv=\frac{1}{4m}\int\Psi^*\left(-i\hbar\nabla-\frac{2e}{c}\mathbf{A}\right)^2\Psi dv. \quad (40)$$

Because Ψ satisfies Eq. (5a), this expression can be written as

$$\frac{1}{4m}\int\Psi^*\left(-i\hbar\nabla-\frac{2e}{c}\mathbf{A}\right)^2\Psi dv=-\int(a|\Psi|^2+b|\Psi|^4+\alpha_{ij}u_{ij}|\Psi|^2)dv. \quad (41)$$

As a result, the free energy (3) can be presented as

$$F=\frac{1}{v}\int\left(\frac{H^2}{8\pi}-\frac{b}{2}\left|\Psi\right|^4+\frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl}\right)dv. \quad (42)$$

This expression generalizes the Abrikosov's one [see Eq. (2)] by taking into account the elastic degrees of freedom. In Ref. 12 it has been reported a similar expression that is erroneous, however (see Sec. V B below).

A. $H \ll H_{c2}$

When treating the fields far from H_{c2}, it is convenient to put $|\Psi|^2=|\Psi_s|^2-h$, where $|\Psi_s|^2=-a/b^*$ [see Eq. (7)] and h represents now the VL contribution. In addition, we present the strain tensor as $u_{ij}=u_{ij}^s+u_{ij}^v$, where $u_{ij}^s=-\alpha_{kl}\lambda_{ijkl}^{-1}|\Psi_s|^2$ and

$$u_{ij}^v=\alpha_{kl}\lambda_{ijkl}^{-1}\langle h\rangle+\frac{1}{2}\sum_{\mathbf{q}\neq 0}[q_iS_k(\mathbf{q})G_{kj}(\mathbf{q})+q_jS_k(\mathbf{q})G_{ki}(\mathbf{q})]h(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}. \quad (43)$$

Thus the equations of equilibrium (14a) and (14b) are satisfied. Substituting these expressions for $|\Psi|^2$ and u_{ij} into Eq. (42), one finds that

$$F=F_s+\frac{1}{v}\int\left(\frac{H^2}{8\pi}+b^*\left|\Psi_s\right|^2h-\frac{b}{2}h^2\right)dv+\frac{1}{2}\sum_{\mathbf{q}}b'(\mathbf{q})|h(\mathbf{q})|^2, \quad (44)$$

where $F_s=-b^*|\Psi_s|^4/2$ and

$$b'(\mathbf{q})=\begin{cases}\alpha_{ij}\alpha_{kl}\lambda_{ijkl}^{-1} & (\mathbf{q}=0),\\ S_i(\mathbf{q})S_j(\mathbf{q})G_{ji}(\mathbf{q}) & (\mathbf{q}\neq 0).\end{cases} \quad (45)$$

Note that, for $\mathbf{q}\neq 0$, the function $b'(\mathbf{q})$ depends on the $\mathbf{q}$ direction only.

Calculating h, we can retain the lowest-order terms in $\hat{\alpha}$, i.e., we can take $h\simeq h_0+h_1$ where h_0 is given by solving Eq. (5a) with $\hat{\alpha}=0$, and h_1 represents the correction to this solution due to the term $\alpha_{ij}u_{ij}^v$ in Eq. (5a):

$$bh_1\simeq\begin{cases}\alpha_{ij}u_{ij}^v & (\text{out of cores}),\\ 0 & (\text{inside cores}).\end{cases} \quad (46)$$

The absence of correction inside the cores follows from the fact that, in these regions, Eq. (5a) can be linearized because of the smallness of the order-parameter modulus (see, e.g., Ref. 15). Let us remark that even when doing so, the vortex cores are taken into account: they act as strain sources.

Substituting these expressions in Eq. (44), and retaining the lowest-order terms, we obtain

$$F=F_s+b^*|\Psi_s|^2\zeta\langle h_0\rangle+\frac{\langle H^2\rangle}{8\pi}-\frac{b}{2}\langle h_0^2\rangle-b\sum_{\mathbf{q}}h_0(-\mathbf{q})h_1(\mathbf{q})+\frac{1}{2}\sum_{\mathbf{q}}b'(\mathbf{q})|h_0(\mathbf{q})|^2, \quad (47)$$

where $\zeta=1+b_0'/b$.

The most important contribution to the two last terms in Eq. (47) arises from $q<\xi^{-1}$.[23] At these $\mathbf{q}$'s, the function $h_1(\mathbf{q})$ can be calculated by taking $bh_1\simeq\alpha_{ij}u_{ij}^v$ in all the regions [see Eq. (46)], so

$$-b\sum_{\mathbf{q}}h_0(-\mathbf{q})h_1(\mathbf{q})\simeq-\sum_{\mathbf{q}}b'(\mathbf{q})|h_0(\mathbf{q})|^2. \quad (48)$$

As a result, the free energy (47) is

$$F\simeq F_s+b^*|\Psi_s|^2\zeta\langle h_0\rangle+\frac{\langle H^2\rangle}{8\pi}-\frac{b}{2}\langle h_0^2\rangle-\frac{1}{2}\sum_{\mathbf{q}}b'(\mathbf{q})|h_0(\mathbf{q})|^2. \quad (49)$$

The last term of this expression represents the strain-induced contribution to the VL energy. The term with $\mathbf{q}=0$ is associated with the homogeneous strains. The elastic constants enter this term through an invariant combination [see Eq. (45)], so it does not depend on the orientation of the VL with respect to the crystal axes. This dependence arises from the terms with $\mathbf{q}\neq 0$.

Let us mention that this formula demonstrates that the elasticity-driven interaction between vortices does not depend on the sample form, unlike to the statement made in Ref. 11. Indeed, such a dependence would mean that contribution to the sum from the region of small q's is essential and comparable with the contribution of the rest of the sum. But the function $h_0(\boldsymbol{\rho})-\langle h_0\rangle$ is a periodic function defined in a finite volume (neglecting the near-surface distortions). Its Fourier spectrum does not contain small q's but has maxima at the nonzero reciprocal-lattice vectors. The form and the size of the sample is reflected in the form and the width of these maxima and nowhere else. In fact, the sums over $\mathbf{q}$'s can be replaced by sums over the reciprocal-lattice vectors ($\mathbf{Q}$) of the VL. Putting

$$h_0(\boldsymbol{\rho}) = \sum_i \tilde{h}_0(\boldsymbol{\rho} - \boldsymbol{\rho}_i), \tag{50}$$

where $\boldsymbol{\rho}_i$ represent the vortex positions, one finds that

$$h_0(\mathbf{q}) = \frac{1}{A} \sum_i \int \tilde{h}_0(\boldsymbol{\rho} - \boldsymbol{\rho}_i) e^{-i\mathbf{q}\cdot\boldsymbol{\rho}} d^2\boldsymbol{\rho} = n\tilde{h}_0(\mathbf{q}), \tag{51}$$

where n is the vortex density, A is the section of the sample perpendicular to the vortex direction, and

$$\tilde{h}_0(\mathbf{q}) \equiv \begin{cases} \int \tilde{h}_0(\boldsymbol{\rho}) e^{-i\mathbf{q}\cdot\boldsymbol{\rho}_i} d^2\boldsymbol{\rho} & (\mathbf{q} = \mathbf{Q}), \\ 0 & (\text{otherwise}), \end{cases} \tag{52}$$

with $\mathbf{Q}$ any of the reciprocal-lattice vectors (note that $A^{-1}\Sigma_i e^{-i\mathbf{q}\cdot\boldsymbol{\rho}_i} = n\,\delta_{\mathbf{qQ}}$). As a result, the last term in Eq. (49) can be written as

$$F_{\text{el}} = -\frac{n^2}{2}\left[b'(0)\tilde{h}_0^2(0) + \sum_{\mathbf{Q}\neq 0} b'(\mathbf{Q}) \left| \tilde{h}_0(\mathbf{Q}) \right|^2 \right]. \tag{53}$$

Let us emphasize that with this expression, one takes into account that both core and non-core regions act as strain sources. It can be straightforwardly illustrated close to H_{c1}. Here, due to the large separation between vortices, the function $\tilde{h}_0$ in Eq. (50) practically coincides with the function associated with one single vortex [see Eq. (12)]. Therefore the function $\tilde{h}_0(\mathbf{Q})$ varies slowly up to $Q \approx \xi^{-1}$ and then rapidly drops to zero. So in Eq. (53) it can be approximated by $\tilde{h}_0(\mathbf{Q}) \simeq \tilde{h}_0(0)$ and naturally splits into core and non-core contributions [see Eqs. (16)]:

$$\tilde{h}_0(0) = \pi(1 + 4\ln\kappa)|\Psi_s|^2 \xi^2/2, \tag{54}$$

limiting the sum over $\mathbf{Q}$'s up to $Q_{\max} \lesssim \xi^{-1}$.

Let us calculate F_{el} explicitly for the isotropic case. In this case one has $\alpha_{ij} = \alpha\delta_{ij}$ and $\lambda_{ijkl} = (K - \frac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where K and μ are the bulk and the shear modulus respectively. Therefore

$$b'(\mathbf{Q}) = \begin{cases} \alpha^2/K & (\mathbf{Q} = 0), \\ \alpha^2/(K_{4/3}) & (\mathbf{Q} \neq 0), \end{cases} \tag{55}$$

where $K_{4/3} = K + 4\mu/3$, and Eq. (53) yields

$$\begin{aligned} F_{\text{el}} &= -\frac{n^2}{2}\tilde{h}_0^2(0)\left(\frac{\alpha^2}{K} + \sum_{\mathbf{Q}\neq 0}^{\mathbf{Q}_{\max}} \frac{\alpha^2}{K_{4/3}} \right) \\ &= -\frac{n^2}{2}\tilde{h}_0^2(0)\left[\frac{4\alpha^2\mu}{3K(K_{4/3})} + \sum_{\mathbf{Q}=0}^{\mathbf{Q}_{\max}} \frac{\alpha^2}{K_{4/3}} \right] \\ &\simeq -\frac{\alpha^2\tilde{h}_0^2(0)}{2(K_{4/3})}\left(\frac{4\mu n^2}{3K} + \frac{n}{\xi^2} \right), \end{aligned} \tag{56}$$

where the sum over discrete $\mathbf{Q}$'s has been replaced by integration ($\Sigma_{\mathbf{Q}} \approx n^{-1}\int d^2\mathbf{Q}$). The term $\propto n$ represents a renormalization of the vortex self-energy, while the term $\propto n^2$ is the elasticity-driven interaction.

In the anisotropic case the sum over $\mathbf{Q}$'s in Eq. (53) also yields a term $\propto n$, which renormalizes the vortex self-energy, and a term $\propto n^2$ which contributes to the elasticity-driven interaction between vortices. In this case, the both term depend on the orientation of the VL with respect to the crystal axes.

Taking into account that $|\Psi_s|^4 b' \sim H_c^2(\Delta K/K)$ and $nH_c\xi^2 \sim B/\kappa$, the elasticity-driven interaction can be estimated as

$$F_{\text{int}} \sim -\frac{(1 + 4\ln\kappa)^2}{\kappa^2}\frac{\Delta K}{K}B^2. \tag{57}$$

As we see, its order of magnitude coincides with what we obtained in Sec. III B from qualitative estimations, as well as with the exact results that we obtained in Sec. IV for the isotropic case. Let us mention that omitting $\ln\kappa$ in these expressions, i.e., omitting the non-core contribution to the vortex-induced strain, they reproduce the previously reported results.[11]

B. $H \approx H_{c2}$

Let us now consider the VL's near H_{c2}. When doing so, it is convenient to use the conventional dimensionless units instead of those defined in Eqs. (10). These conventional units can be obtained from Eqs. (10) by replacing $|\Psi_s|^2$ with $-a/b$.

Following Kogan[24] one can easily obtain, now from the equations of equilibrium (5), the so-called Abrikosov identities (see also Refs. 14,15, and 25). In presence of strain they read

$$H_z = H_0 - \frac{\omega}{2\kappa}, \tag{58a}$$

$$\frac{\kappa - H_0}{\kappa}\langle\omega\rangle + \frac{1 - 2\kappa^2}{2\kappa^2}\langle\omega^2\rangle - \alpha_{ij}\langle u_{ij}\omega\rangle = 0. \tag{58b}$$

Here H_0 is a constant and ω is the squared modulus of the function Ψ, which is the solution of linearized equations (5a) and (5b) ($\Psi = \sqrt{\omega}e^{i\chi}$). Bearing in mind that the magnetic induction is $B = \langle H_z\rangle = H_0 - \langle\omega\rangle/(2\kappa)$, from Eqs. (58) one can also obtain the following relationship:

$$\langle\omega\rangle = \frac{2\kappa(\kappa - B)}{\tilde{\beta}_A - \beta_e}, \tag{59}$$

where $\beta_e = -2\kappa^2\alpha_{ij}\langle u_{ij}\omega\rangle/\langle\omega\rangle^2$ and $\tilde{\beta}_A = 1 + (2\kappa^2 - 1)\beta_A$, with $\beta_A = \langle\omega^2\rangle/\langle\omega\rangle^2$.

Bearing in mind that from the Abrikosov identity (58a) it follows that $\langle H^2\rangle = B^2 + (\langle\omega^2\rangle - \langle\omega\rangle^2)/(4\kappa^2)$, the free energy (42) can be rewritten as

$$F = B^2 - \frac{\tilde{\beta}_A}{4\kappa^2}\langle\omega\rangle^2 + \frac{1}{2}\lambda_{ijkl}\langle u_{ij}u_{kl}\rangle. \tag{60}$$

Expressing u_{ij} in the form (13) and taking into account that $|\Psi|^2=\omega$, one finds that

$$\epsilon_{ij}=-\alpha_{kl}\lambda^{-1}_{ijkl}\langle\omega\rangle,$$

$$u_i(\mathbf{q})=iS_j(\mathbf{q})G_{ji}(\mathbf{q})\omega(\mathbf{q}), \tag{61a}$$

where $\omega(\mathbf{q})$ is the Fourier transform of the function ω. The last term of Eq. (60) is now

$$\begin{aligned}\frac{1}{2}\lambda_{ijkl}\langle u_{ij}u_{kl}\rangle &= \frac{1}{2}\left[\lambda_{ijkl}\epsilon_{ij}\epsilon_{kl}+\sum_{\mathbf{q}\neq 0}G^{-1}_{ij}(\mathbf{q})u_i(\mathbf{q})u_j(-\mathbf{q})\right]\\ &=-\frac{1}{2}\left[\lambda_{ijkl}\lambda^{-1}_{mnkl}\alpha_{mn}\epsilon_{ij}\langle\omega\rangle+i\sum_{\mathbf{q}\neq 0}G^{-1}_{ij}(\mathbf{q})G_{kj}(\mathbf{q})S_k(\mathbf{q})u_i(\mathbf{q})\omega(-\mathbf{q})\right]\\ &=-\frac{1}{2}\left[\alpha_{ij}\epsilon_{ij}\langle\omega\rangle+i\sum_{\mathbf{q}\neq 0}S_i(\mathbf{q})u_i(\mathbf{q})\omega(-\mathbf{q})\right]=-\frac{1}{2}\alpha_{ij}\langle u_{ij}\omega\rangle=\frac{\beta_e}{4\kappa^2}\langle\omega\rangle^2.\end{aligned} \tag{62}$$

The free energy (60) can be written as

$$F=B^2-\frac{\tilde{\beta}_A-\beta_e}{4\kappa^2}\langle\omega\rangle^2=B^2-\frac{(\kappa-B)^2}{\tilde{\beta}_A-\beta_e}. \tag{63}$$

Let us mention that the form of this expression for the free energy differs substantially from that reported by Miranović *et al.* in Ref. 12. The elasticity-driven interaction term that given by Eq. (63) is several times smaller than the corresponding one in Ref. 12. The validity of Eq. (63) can be checked by noting that it reproduces the isotropic case [see expression (II) in Eq. (26)]. In contrast, the expression reported in Ref. 12 does not (see Sec. IV C). The reason is that it is obtained from an expression analogous to Eq. (42), but erroneous [Eq. (20) of Ref. 12]. Using Eqs. (16) and (12) of Ref. 12 in Eq. (20) of the same reference such an expression reads

$$F=\frac{1}{v}\int\left(\frac{H^2}{8\pi}-\frac{b}{2}|\Psi|^4+\alpha_{ij}u_{ij}|\Psi|^2+\frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl}\right)dv. \tag{64}$$

One can see here that the term $\alpha_{ij}u_{ij}|\Psi|^2$ is taken into account twice: one time explicitly and another time implicitly in the term $-b|\Psi|^2/2$ which arises as a result of the integration by parts performed at the beginning of this section.

VI. CONCLUDING REMARKS

We have revised the contribution to the VL energy which is due to the vortex-induced strains, showing that essential corrections to the previous calculations are needed. The most important one is connected with the fact that, in high-κ superconductors, not only do the vortex cores induce strains in a significant way; there also exists a significant contribution associated with the non-core regions which are, in fact, the most important ones for the VL energies at low fields ($H \ll H_{c2}$). As a result of the proper inclusion of all strain sources, the strength of the elasticity-driven interaction between vortices increases by a factor up to $\sim \ln^2\kappa$ compared with the previously reported ones.

It is known since long ago that the observed correlations between VL's and crystal lattices in dirty superconductors cannot be explained without the elasticity-driven interaction between vortices.[10] This interaction has been proved to be important in clean superconductors also. For example, the VL's observed in Nb Se_2 do not correspond to the minimum of the London energy. In Ref. 11 Kogan *et al.* showed that the difference in the London energies of the two possible competing structures is smaller than the difference in the energies of the corresponding elasticity-driven interactions that they calculated. As we have mentioned, Kogan *et al.* underestimated the elasticity-driven interaction between vortices because they assumed that only the vortex cores induce strain but, even doing so, they pointed out the importance of this interaction in $NbSe_2$. In fact this is even more important as we have shown in the present work, which should be taken into account especially in those cases in which previous estimates concluded that the London energy was the most important one.

V_3Si might provide an example in which the latter case takes place. In Ref. 26 it was claimed that in V_3Si the contribution to the VL energy which is due to the (underestimated) elasticity-driven interactions between vortices can be neglected compared to the contribution due to the nonlocal corrections to the London energy. But bear in mind that (i) the order of magnitude of these two contributions is the same, as it was shown in Ref. 11 considering the vortex cores as the only sources of strains, and (ii) the elasticity-driven interaction is considerably stronger than it was reported, as we have shown in this paper. So it is quite probable that in V_3Si, as well as in other superconductors with large κ, this elasticity-driven interaction between vortices is not only comparable, but even more important than the nonlocal corrections to the London energy.

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APPENDIX

Let us check that the expressions in Eq. (21) match one each other at the boundaries of the magnetic-field regions defined therein, with reasonable accuracy. As we already mentioned, this matching permits us to study the elastic effects in the whole region of the vortex state of isotropic superconductors.

Recall first that the magnetic induction, as a function of the magnetic field, is given by[22]

$$B\simeq\begin{cases}\dfrac{2\phi_0}{\sqrt{3}\lambda_L^2}\ln^{-2}\left[\dfrac{3\phi_0}{4\pi\lambda_L^2(H-H_{c1})}\right] & \text{(I)},\\[2ex] H-H_{c1}+\dfrac{\phi_0}{8\pi\lambda_L^2}\left\{\ln\left[\dfrac{4\pi\lambda_L^2(H-H_{c1})}{\phi_0}\right]+\tilde{\gamma}\right\} & \text{(I–II)},\\[2ex] H-\dfrac{H_{c2}-H}{(2\kappa^2-1)\beta_A} & \text{(II)},\end{cases}\tag{A1}$$

where $\tilde{\gamma}=2(1-\gamma)$.

For $H=\zeta_1 H_{c1}$, where $\zeta_1\gtrsim 1$ is a numerical factor of order unity, according to the expressions (I–II) and (II) in Eq. (A1) one has $B\simeq\zeta_1 H_{c1}$. Therefore, taking into account the relationship (23), one finds that

$$F_{\rm VL}\simeq\begin{cases}\dfrac{\zeta_1 H_{c1}^2}{4\pi} & \text{(I)},\\[2ex] \dfrac{\zeta_1 H_{c1}^2}{8\pi}\left(\zeta_1+\dfrac{\ln[8\pi(\nu\kappa)^2/(\sqrt{3}\zeta_1\ln\kappa)]}{\ln\kappa}\right) & \text{(I–II)}.\end{cases}\tag{A2}$$

As we can see, the value of the free energy given by both expressions is $F_{\rm VL}\simeq H_{c1}^2/(4\pi)$ up to logarithmic corrections.

For $H=\zeta_2 H_{c2}$, where $\zeta_2\lesssim 1$ is a new numerical factor, according to the expressions (I–II) and (II) in Eq. (A1) and the relationship (22) one has $B\simeq\zeta_2 H_{c2}$. Therefore, taking into account the relationship (23), one finds that

$$F_{\rm VL}\simeq\begin{cases}\dfrac{\zeta_2 H_{c2}^2}{8\pi}\left(\zeta_2+\dfrac{\ln[4\pi\nu^2/(\sqrt{3}\zeta_2)]}{2\kappa^2}\right) & \text{(I–II)},\\[2ex] \dfrac{H_{c2}^2}{8\pi}\left(\zeta_2^2-\dfrac{(1-\zeta_2)^2}{2\kappa^2}\right) & \text{(II)}.\end{cases}\tag{A3}$$

Here we see that these two expressions give values of the free energy that match each other: $F_{\rm VL}\simeq H_{c2}^2/(8\pi)$.

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[23] Note that, in principle, d^{-1}, λ_L^{-1}, and ξ^{-1} are the characteristic q's of the functions $h_{0,1}(\mathbf{q})$ (d is the distance between vortices). Close to H_{c1} they are such that $d^{-1}\ll\lambda_L^{-1}\ll\xi^{-1}$. For intermediate fields, however, λ_L is not a characteristic length of the VL because of the overlap between vortices. The intermediate region is defined by the condition $d^{-1}\ll\xi^{-1}$.
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