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LATTICE DYNAMICS AND PHASE TRANSITIONS

Influence of Striction on Soliton Interaction in Crystals

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Abstract—Striction-mediated attraction of domain walls, solitons in incommensurate phases, and Abrikosov vortices in superconductors are considered. It is shown (a) that it is this type of attraction that can be responsible for a soliton-density jump in lock-in transitions and (b) that the strain-induced vortex interaction in superconductors with a high Ginzburg–Landau parameter is higher by one or two orders of magnitude than was assumed earlier. © *2005 Pleiades Publishing, Inc.*

INTRODUCTION

The influence of long-range elastic interactions on the properties of modulated phases, which are the regular structures of solitons, has long attracted attention in connection with the study of the properties of various systems such as dielectrics possessing structurally incommensurate phases [1] and second-order superconductors with Abrikosov-vortex lattices [2, 3]. Usually, this problem in incommensurate phases was studied with the aim to establish whether a lock-in transition is continuous or the soliton density at the transition point changes in a jumpwise manner. As a rule, the mechanisms that could give rise to a soliton-density jump were examined with no allowance for the striction effect. At the same time, in studies of vortex-lattice orientations in crystals, the striction-mediated interaction in superconductors was often considered as one of the main types of interactions. Such interaction was usually calculated based on a simplified model under the assumption that the elastic strains inducing vortex interactions are due only to vortex cores. However, it turned out that the interactions in these studies were considerably underestimated. Therefore, the present study is dedicated to the consideration of these problems.

At the beginning, the striction effect is considered on the simplest example of domain walls described by the one-dimensional distribution of a one-component order parameter. This example allows us to reveal the characteristic features of this interaction and to evaluate it for different types of domain walls. Then, following the concepts stated in [4, 5], we calculate the striction-mediated attraction of two-dimensional solitons in incommensurate phases and Abrikosov vortices in superconductors. Some computations are performed by a method somewhat different from the method used in the studies cited above, and some of the results obtained are considered in more detail.

A POLYDOMAIN CRYSTAL

Consider a polydomain structure described by a spatially inhomogeneous distribution of a one-component order parameter $\eta(x)$. The order parameter inside a domain wall is inhomogeneous and, at a certain point, goes to zero. Variation of the order parameter inside the wall should change the crystal strains in such a way that the temperature variation in a certain layer of a crystal undergoes no phase transitions. In this case, the relief at the site of the wall intersection by the surface should be distorted in conformity with relaxation of elastic stresses in the vicinity of the surface. It is natural that the strain and order-parameter distributions in the vicinity of the surfaces and in the crystal bulk are different. The strain distributions in the bulk can be calculated under the condition of zero bulk stresses. Then, in order to obtain the exact solution of the problem, one has to introduce some additional imaging forces having the zero average values at the surface. Since the problem is of a periodic nature, these forces should have a periodic distribution along the surfaces. As is well known [6], these forces give rise to additional strains decreasing in the crystal depth within a characteristic length of the order of the period of a surface-force distribution. Therefore, the contribution of the near-surface distortions to the energy of this regular structure is rather small because of the small ratio of the structure period to the crystal size. However, the distributions in the crystal bulk we are interested in are one-dimensional. It should be indicated that the solution of this elastic problem at the given one-dimensional distribution of the strain sources and arbitrary anisotropy was obtained in [7]. We are interested in the solution of a

more complicated problem in which the spatial distributions of strains and order parameter vary self-consistently.

Consider the case of an elastically isotropic medium and analyze the anisotropic case of an example of a more complicated domain-wall structure in an incommensurate phase. Represent the energy per volume unit of the system, f, in the form

$$f = \frac{1}{V}\int\left[\frac{1}{2}A\eta^2 + \frac{1}{4}B\eta^4 + \frac{1}{2}D\left(\frac{d\eta}{dx}\right)^2 + r\eta^2 u_{ll} + \mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right)^2 + \frac{K}{2}u_{ll}^2\right]dv, \quad (1)$$

where V is the sample volume, η is the order parameter varying along x, u_{ik} is the strain tensor, μ is the shear modulus, K is the bulk modulus, and $A = A_T(T - T_c) < 0$; i.e., the phase has a low symmetry.

For a homogeneous system, we have

$$\eta_e^2 = -\frac{A}{B'}, \quad u_{lle} = -\frac{r}{K}\eta_e^2, \quad f_{\text{hom}} = -\frac{1}{4}\frac{A^2}{B'}, \quad (2)$$

where $B' = B - 2r^2/K$.

Following [8], we start calculating the energy of a polydomain structure with the solution of the elastic problem. For one-dimensional strain distributions (along the x axis), the corresponding compatibility conditions have the form

$$\frac{d^2u_{zz}}{dx^2} = 0, \quad \frac{d^2u_{yy}}{dx^2} = 0, \quad \frac{d^2u_{yz}}{dx^2} = 0. \quad (3)$$

With due regard for the problem symmetry in the yz plane, only the following solutions of above equations are possible: $u_{zz} = u_{yy} = \tilde{u}$ and $u_{yz} = 0$, where $\tilde{u}$ is a constant.

The equations of the local elastic equilibrium have the form

$$\frac{d\sigma_{xx}}{dx} = 0, \quad \frac{d\sigma_{xy}}{dx} = 0, \quad \frac{d\sigma_{xz}}{dx} = 0, \quad (4)$$

where, in accordance with Eq. (1), elastic stresses have the form

$$\sigma_{ij} = \left(K - \frac{2}{3}\mu\right)u_{ll}\delta_{ij} + 2\mu u_{ij} + r\eta^2\delta_{ij}. \quad (5)$$

Moreover, in the absence of any external stresses, the σ_{ij} values averaged over the bulk should be equal to zero [9]:

$$\langle\sigma_{ij}\rangle = 0. \quad (6)$$

From Eqs. (4)–(6), we have

$$\sigma_{xx} = \left(K + \frac{4}{3}\mu\right)u_{xx} + \left(K - \frac{2}{3}\mu\right)(u_{yy} + u_{zz}) + r\eta^2 = 0, \quad (7)$$

$$\sigma_{xy} = 2\mu u_{xy} = 0, \quad \sigma_{xz} = 2\mu u_{xz} = 0. \quad (8)$$

As a result, the nonzero components of strain are

$$u_{xx} = -\left[\left(K - \frac{2}{3}\mu\right)2\tilde{u} + r\eta^2\right]\Big/\left(K + \frac{4}{3}\mu\right), \quad u_{zz} = u_{yy} = \tilde{u}. \quad (9)$$

Substituting these solutions into Eq. (1), we arrive at the free-energy density in the form

$$f = \frac{1}{2L}\int_{-L}^{L}\left[\left(\frac{A + 2r'\tilde{u}}{2}\right)\eta^2 + \frac{1}{4}B''\eta^4 + \frac{1}{2}D\left(\frac{d\eta}{dx}\right)^2 + \frac{K'}{2}\tilde{u}^2\right]dx, \quad (10)$$

where $2L$ is the sample dimension along the x axis, $r' = 4r\mu/(K + 4\mu/3)$, $K' = 12K\mu/(K + 4\mu/3)$, and $B'' = B - 2r^2/(K + 4\mu/3)$. Then, $\tilde{u}$ may be considered as a certain parameter which, similar to $\eta(x)$, may be determined by minimizing the free energy described by Eq. (10). Assuming that the distance l between the walls is much larger than the wall width, r_c, we may represent the solution for a polydomain structure (with the accuracy of exponentially small corrections) as a sum of the solutions $\sum_m \eta_1(x + ml)$ corresponding to isolated walls

$$\eta(x) = \sum_m \frac{(-A - 2r'\tilde{u})^{1/2}}{B''^{1/2}} \times \tanh\left(\frac{x - ml}{2^{1/2}D^{1/2}/(-A - 2r'\tilde{u})^{1/2}}\right). \quad (11)$$

Substituting Eq. (11) into Eq. (10), we obtain

$$f = -\frac{(A + 2r'\tilde{u})^2}{B''} + \frac{K'}{2}\tilde{u}^2 + \frac{2^{3/2}D^{1/2}(-A - 2r'\tilde{u})^{3/2}}{3B''}n, \quad (12)$$

where $n = l/(2L)$ is the wall concentration. If $n \ll 1$, then, minimizing Eq. (12) with respect to $\tilde{u}$, we obtain

$$f = -\frac{A^2}{4B'} + \frac{2^{3/2}D^{1/2}|A|^{3/2}B''^{1/2}}{3B'^{3/2}}n - \frac{16\mu r^2 D|A|B''^{1/2}}{3\tilde{K}KB'^2}n^2, \quad (13)$$

where $\tilde{K} = K + 4\mu/3$. The second term in the right-hand side of Eq. (13) corresponds to the sum of wall self-energies, whereas the third term corresponds to wall attraction.

Consider the interaction effect in more detail. It should be noted that Eq. (6) yields the ratio $\langle u_{ll}\rangle =$

$-r\langle\eta^2\rangle/K$, which, together with Eq. (9), gives rise to the equalities $u_{zz} = u_{yy} = \tilde{u} = -r\langle\eta^2\rangle/3K$. Then, Eqs. (7), (9), and (11) yield longitudinal stresses as

$$\sigma_{yy} = \sigma_{zz} = \frac{2\mu r(\eta^2 - \langle\eta^2\rangle)}{\tilde{K}} = \frac{2\mu r\eta_e^2}{K}\sum_m\left\{-\frac{1}{\cosh^2\dfrac{x-ml}{2r_c}} + \frac{r_c}{l}\right\}, \quad (14)$$

where $r_c = (DB')^{1/2}/(2|A|B'')^{1/2}$ is the correlation radius. It is seen from Eq. (14) that the walls give rise to longitudinal stresses not only in the regions of their localization but also in the whole crystal bulk, which results in the wall interaction.

INCOMMENSURATE PHASE

In the case of a multicomponent order parameter, striction corresponds to coupling between strain and the squared modulus of the order parameter. Moreover, the striction-mediated interaction strongly depends on the wall type. Thus, in the case of Bloch walls, the modulus of the order parameter, ρ, is constant in the region of wall localization; therefore, there is no wall interaction: the longitudinal stresses have zero values since $\rho^2 - \langle\rho^2\rangle \equiv 0$ (see Eq. (14)). For quasi-Bloch walls characteristic of incommensurate (IC) phases of type I (with the Lifshitz invariant in the free-energy expansion) with weak anisotropy in the space of order-parameter components, the local quantity $\rho^2 - \langle\rho^2\rangle$ has a very low nonzero value [1]. However, in this case, the wall width is of the order of a reciprocal wave vector (q_0) of the structure at the point of the transition "normal (*N*) phase–IC phase," whose typical value is of the order of $10^{-2}d_{at}^{-1}$ (where d_{at} is the interatomic distance). This signifies that, although the additional dilatation is rather small, it arises in extended regions because of a considerably increased wall interaction. It should be emphasized that the case of weak anisotropy has drawn great interest because the continuity of the lock-in transition in systems having no long-range interactions was rigorously proven [10]. This case is described in detail elsewhere [4]. Here, we only derive the basic relationships for the energy of soliton interaction in the vicinity of such a transition by a somewhat modified method.

In the simplest case, an IC phase is described by a one-dimensional modulation (along the *x* axis) of a certain two-component order parameter ($\eta_1 = \rho\cos\varphi$ and $\eta_2 = \rho\sin\varphi$, where ρ is the amplitude and φ is the phase of the order parameter). This order parameter describes lowering of the symmetry in the transition from the normal phase to the low-symmetric commensurate C phase. Then, free energy may be represented as

$$f = \frac{1}{V}\int\Big[\alpha\rho^2 + \beta\rho^4 + \gamma\rho^m\cos(m\varphi) - \sigma\rho^2\frac{\partial\varphi}{\partial x} + \delta\rho^2(\nabla\varphi)^2 + \delta(\nabla\rho)^2 + r_{ij}\rho^2u_{ij} + \frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl}\Big]dv. \quad (15)$$

Here $\alpha = \alpha_T(T - \theta)$ and *m* is the anisotropy order ($m \geq 3$).

At the temperature $T_i > \theta$ determined by the conditions $\alpha_0 \equiv \alpha_T(T_i - \theta) = \sigma^2/4\delta$, the *N* phase undergoes a second-order transition to the IC phase, whose structure is described by a one-harmonic distribution of the order parameter ($\eta_1 = \rho\cos(q_0x)$, $\eta_2 = \rho\sin(q_0x)$) with the wave vector $q_0 \equiv \sigma/(2\delta)$. Because of anisotropy of the space of order-parameter components, the wave vector decreases with lowering of temperature, and the structure of the IC phase is transformed from a harmonic one into a domain-like one. Therefore, a lock-in (IC–C) transition undergone at a certain temperature may be considered as a transition leading to disappearance of domain walls.

As earlier, solving the elastic problem, we ignore near-surface distortions formed in a finite sample and consider, first, an elastically isotropic medium by setting that $\lambda_{ijkl} = [K - (2/3)\mu]\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and $r_{ij} = r\delta_{ij}$. In this case, Eqs. (3)–(9) remain valid if we make the change $\eta \longrightarrow \rho$. Then, using the notation $u_{zz} = u_1$, we obtain instead of Eq. (10) the following equation:

$$f = \frac{1}{2L}\int\Big[\alpha(u_1)\rho^2 + \beta''\rho^4 + \gamma\rho^m\cos(m\varphi) - \sigma\rho^2\frac{\partial\varphi}{\partial x} + \delta\rho^2(\nabla\varphi)^2 + \delta(\nabla\rho)^2\Big]dx + \frac{K'u_1^2}{2}, \quad (16)$$

where $\alpha(u_1) = \alpha + r'u_1$, r' and K' have the same values as in Eq. (10), $\beta'' = \beta - r^2/(2\tilde{K})$, and $\tilde{K} = K + 4\mu/3$.

At a fixed value *u*, the distributions of the order parameter and free energy of the IC phase in the vicinity of the IC–C transition may be represented as the expansions in the anisotropy parameter $\varepsilon_m = -(m\pi^2/2^4)[\alpha_0/\alpha_c]$, where $\alpha_c = -2\beta''[\sigma^2/(2^5\gamma\delta)]^{2/(m-2)}$ [10]. In a lower approximation (approximation of a constant amplitude), the energy of the IC phase, being a function of the soliton density (*n*), has the form [1]

$$f = -\beta''\rho^4(u_1) + E(u_1)n + 4J(u_1)n\exp(-mp(u_1)/(2n) + \frac{K'}{2}u_1^2). \quad (17)$$

The coefficients in Eq. (17) are expressed in terms of the squared amplitude of the order parameter, $\rho^2(u_1) = -\alpha(u_1)/(2\beta'')$, and the wave number, $q_0 = \sigma/(2\delta)$, of the

IC structure at the point of the N–IC transition as follows:

$$p^2(u_1) = \frac{2\gamma}{\delta}\rho^{m-2}(u_1), \quad (18)$$

$$J(u_1) = \frac{4\sigma\rho^2(u_1)p(u_1)}{mq_0}, \quad (19)$$

$$E(u_1) = \frac{2\sigma\rho^2(u_1)}{mq_0}[2p(u_1) - \pi q_0]. \quad (20)$$

In this case, it follows from equations analogous to Eqs. (6)–(9) that $u_{zz} = u_{yy} = u_1 = -r\rho^2(u_1)/(3K)$. Substituting this relationship into Eq. (16), we see that, in the approximation of a constant amplitude, strains result only in the renormalization of the coefficient before ρ^4; i.e., $\beta \longrightarrow \beta' = \beta - r^2/(2K)$, and, therefore, $u_1 = r\alpha/(6K\beta') = u_c/3$ and $\rho^2(u_c) = -\alpha/(2\beta')$. Thus, in this approximation, solitons do not interact. In the next approximation with respect to the anisotropy parameter, two corrections appear: a spatially inhomogeneous correction to $\rho^2(u_c)$ denoted as $\rho_1^2(x)$ [1] and the corresponding correction to energy (17). Now, the equations of elastic equilibrium yield

$$u_{zz} - u_1 = u_{yy} - u_1 = \varepsilon_1,$$
$$u_{xx} - u_1 = -\left[\left(K - \frac{2}{3}\mu\right)2\varepsilon_1 + r\rho_1^2(x)\right]\Big/\left(K + \frac{4}{3}\mu\right), \quad (21)$$

where the additional strain $\varepsilon_1 = -r\langle\rho_1^2(x)\rangle/(3K)$ caused by appearance of solitons is proportional to the soliton density n. In the vicinity of the IC–C transition, $n \ll 1$; therefore, $\varepsilon_1 \ll u_c$. Then, we may expand Eq. (17) and minimize the result with respect to ε_1. As a result, we have $\varepsilon = -nr'E\beta''/K\beta'$ and the free energy of solitons has the form

$$\delta f \approx a_1 n - a_3 n^2 + a_2 n\exp(-N/n), \quad (22)$$

where $a_1 = E_1[\alpha(u_c) - \alpha_c]$; $a_3 = (r')^2E_1^2\beta'/(2K'\beta')$; $E_1 = (2 - m)\pi\sigma/(4m\beta'')$; $N = m\pi q_0/4$; and $a_2 = -4\pi\sigma\alpha_c/(m\beta'')$ is the energy of an isolated soliton, which goes to zero at the temperature determined by the condition $\alpha(u_c) = \alpha_c$. The term $-a_3n^2$ describes the soliton attraction and gives rise to a jump in the soliton density at the point of the IC–C transition.

Note that the main term of the soliton-energy expansion in an anisotropy parameter $\alpha - \tilde{\alpha}_c$ for the case $m = 4$ was calculated in [10]. In our notation, it has the form

$$\delta f = \frac{\pi^2\alpha_0(\alpha - \tilde{\alpha}_c)}{2\beta\ln\left|\dfrac{\alpha - \tilde{\alpha}_c}{4\alpha_c}\right|}. \quad (23)$$

The $\tilde{\alpha}_c$ value was calculated with a higher accuracy than α_c. Introducing the dependence of α on u into Eq. (23), we may, as earlier, calculate the value of soliton attraction. It turns out that the same (within the change $\alpha_c \longleftrightarrow \tilde{\alpha}_c$) result may also be obtained by minimizing Eq. (22) with respect to soliton density with a subsequent singling out of the main term of expansion in $\alpha - \alpha_c$ in the expression thus obtained.

Now, we show that the expression for free energy in the vicinity of the IC–C transition for anisotropic systems has the same form as Eq. (22) and that the coefficients in this expression may be obtained by the corresponding renormalization. We vary the initial expressions for free energy, Eqs. (15) and (16), with respect to the elastic degrees of freedom, and then compare the functionals thus obtained. Equation (15) should be varied separately for homogeneous strains ($\langle u_{ij}\rangle$) and inhomogeneous elastic displacements (u_i), which represent the independent degrees of freedom. It is convenient to pass to the following Fourier representation

$$u_{ij} = \langle u_{ij}\rangle + \frac{i}{2}\sum_{k\neq 0}[k_iu_j(\mathbf{k}) + k_ju_i(\mathbf{k})]\exp(i\mathbf{kr}), \quad (24)$$

where, in virtue of one-dimensionality of the problem under consideration, $\mathbf{k} = (k_x, 0, 0)$. For simplicity, we limit our consideration to often-encountered systems described by the symmetry class D_{2h}. Then, minimizing Eq. (15) with respect to the elastic degrees of freedom in the $\mathbf{k}$ space, we obtain from the last two terms

$$\delta f = -\frac{1}{2}r_{ij}r_{kl}\lambda^{-1}_{ijkl}f_0^2 - \frac{r_{33}^2}{2\lambda_{3333}}\sum_{k_x\neq 0} f_{k_x}f_{-k_x}, \quad (25)$$

where f_{k_x} is the Fourier component of the function $f = \rho^2$ and λ^{-1}_{ijkl} is the tensor reciprocal to the tensor λ_{ijkl}. Returning to the real space in Eq. (25), we obtain

$$\delta f = -\frac{1}{2}\left[r_{ij}r_{kl}\lambda^{-1}_{ijkl} - \frac{r_{33}^2}{\lambda_{3333}}\right]\langle\rho^2\rangle^2 - \frac{r_{33}^2}{2\lambda_{3333}}\langle\rho^4\rangle, \quad (26)$$

where $\langle\ldots\rangle$ indicates averaging over the bulk. Here, the last term is understood as the renormalization of the term $\beta\langle\rho^4\rangle$ in Eq. (15).

In turn, minimization of the elastic contribution in the isotropic case yields

$$\delta f = -\frac{2r^2\mu}{3K\tilde{K}}\langle\rho^2\rangle^2 - \frac{r^2}{2\tilde{K}}\langle\rho^4\rangle. \quad (27)$$

Comparing Eqs. (26) and (27), we see that the free-energy functionals which are determined by the distribution of the order parameter alone have the same form in both cases and differ only by their coefficients. It follows that, minimizing these functionals with respect to the order-parameter distribution, one arrives at the

same results for free energy within the accuracy of the following replacements:

$$\beta - r^2/(2\tilde{K}) \longrightarrow \beta - r_{33}^2/(2\lambda_{3333}),$$
$$4r^2\mu/(3K\tilde{K}) \longrightarrow r_{ij}r_{kl}\lambda_{ijkl}^{-1} - r_{33}^2/\lambda_{3333}. \quad (28)$$

When considering the IC–C transition in the systems characterized by weak anisotropy, one has also to take into account some other interactions [4]. For displacive phase transitions, the most important of which is described by the dependence of the Lifshitz invariant on strain. The estimates made in [4] show that if anisotropy is not too weak ($10^{-2} < \varepsilon_m < 1$) the striction contribution prevails in attraction. However, in ferroelectric systems of the order–disorder type, one more mechanism may play an important role: attraction due to thermal domain-wall bending.

As a result of attraction, an IC–C transformation should be a first-order transition. Expression (22) with the renormalized coefficients allows one to determine the basic transition characteristics: the transition temperature, the temperature of maximum supercooling, the soliton density at the transition point, the latent heat of transition, and the anomaly in heat capacity. It should be indicated that the anomalous part of heat capacity varies according to the Curie–Weiss law and diverges at the point of maximum supercooling.

In the cases of pronounced anisotropy or an IC phase of type II (the Lifshitz invariant is forbidden by the symmetry of the normal phase), the striction-mediated interaction of solitons may be evaluated using the first term on the right-hand side of Eq. (26). This interaction equals $n^2(r^2/\tilde{K})(\Delta\eta^2 r_c)^2$, where $\Delta\eta^2$ is the squared change in the amplitude of the order parameter in a wall of width r_c.

MIXED STATE IN A SUPERCONDUCTOR

The effect of striction-mediated attraction on the properties of vortices in second-order superconductors was first considered in connection with vortex pinning at defects. Much later, it was considered in connection with its influence on the orientation of vortex structures relative to the crystal lattice (see references in [3, 5]).

As in the studies of the thermodynamics of a vortex lattice, in general, when analyzing the above effects, one usually singles out two regions where the external magnetic field is either close or not too close to the upper critical field H_{c2}. To describe the two-dimensional spatial distributions of a complex order parameter $\Psi = \rho\exp(i\phi)$ in these regions, two qualitatively different approximations are used [11]. In the vicinity of H_{c2}, the basic periods of the vortex lattice are close to the correlation radius ξ and the distribution of the order-parameter modulus in the regions between vortices is inhomogeneous. In the fields not too close to H_{c2}, the distances between vortices considerably exceed ξ. In the conventionally used London approximation, it is assumed that the order-parameter modulus varies only in the cores of vortices with radii $r \sim \xi$. Since, in fact, the variation of the squared modulus of the order parameter caused by vortex appearance describes the distribution of the striction-mediated strain sources, it was assumed that, in the applicability range of the London approximation, the role of these strain sources is played by vortex cores. Following [5], consider the effects of a long-range elastic action in an approximation more accurate that the London approximation in the fields $H \ll H_{c2}$. In other words, we take into account the change in the order-parameter modulus not only in the vortex core but also in the surrounding noncore region limited by the penetration depth λ much larger than ξ in superconductors with a pronounced Ginzburg–Landau parameter ($\kappa = \lambda/\xi$).

We proceed from the Ginzburg–Landau free-energy expansion with allowance for its dependence on elastic strains:

$$f = \frac{1}{V}\int\left[\frac{H^2}{8\pi} + a|\Psi|^2 + \frac{b}{2}|\Psi|^4\right. \quad (29)$$
$$\left. + \frac{1}{4m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Psi\right|^2 + r_{ij}|\Psi|^2 u_{ij} + \frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl}\right]dv,$$

where Ψ is the order parameter corresponding to the transition to the superconducting state, **A** is the vector potential, and H is the magnetic field.

The equilibrium equations have the form

$$\left[a + b|\Psi|^2 + r_{ij}u_{ij} + \frac{1}{4m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)^2\right]\Psi = 0, \quad (30)$$

$$\nabla\mathbf{H} = \frac{4\pi e}{mc}\left[\frac{\hbar}{2i}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{2e}{c}|\Psi|^2\mathbf{A}\right], \quad (31)$$

$$\frac{\partial}{\partial x_j}(\lambda_{ijkl}u_{kl} + r_{ij}|\Psi|^2) = 0, \quad (32)$$

$$\lambda_{ijkl}\langle u_{kl}\rangle + r_{ij}\langle|\Psi|^2\rangle = 0, \quad (33)$$

where, as earlier, $\langle\ldots\rangle$ indicates averaging over the bulk.

Relationship (33) describes the result of free-energy variation over homogeneous strains and may be regarded as the necessary condition for absence of any homogeneous external stresses. As in the case of a regular soliton structure in an incommensurate phase considered above, we ignore the near-surface vortex-lattice distortions propagating into the crystal bulk for distances comparable with the period of this lattice.

From Eqs. (30) and (33), we obtain the spontaneous values in the homogeneous superconducting state:

$$|\Psi_s|^2 = -a/b^*, \quad (34)$$

$$u_{ij}^s = -r_{kl}\lambda_{ijkl}^{-1}|\Psi_s|^2 = ar_{kl}\lambda_{ijkl}^{-1}/b^*, \quad (35)$$

where $b^* = b - r_{ij}r_{kl}\lambda^{-1}_{ijkl}$.

To calculate the vortex interactions induced by elastic strains in an isotropic medium, consider the limit of an infinite shear modulus. In this limit, only homogeneous dilatation u exists, which considerably simplifies the solution of the elastic problem. Varying Eq. (29) with respect to the elastic degrees of freedom (homogeneous strains and two-dimensional inhomogeneous displacements) of an elastically isotropic medium with a finite and infinite shear moduli, one can show that all the data for the finite μ may be obtained from the corresponding relationships for the limiting case $\mu = \infty$ after the following renormalization: $b \longrightarrow b - r^2/\tilde{K}$ and $r^2/K \longrightarrow (r^2/K)[4\mu/(3\tilde{K})]$.

In the case $\mu = \infty$, the two last terms of the free-energy expansion (29) have the form $r^2|\Psi|^2u + Ku^2/2$. Strain u may be regarded as a variation parameter. Since this parameter modifies the coefficient before $|\Psi|^2$, we may introduce the notation $a(u) = a + ru_s + r\varepsilon$, where, in accordance with Eq. (35), $u_s = ra/b^*$, $b^* = (b - r^2/K)$, and ε is the vortex-induced strain. Then, using the results obtained in [11, 12], we may represent the free-energy density of the vortex lattice as a function of magnetic induction B ($B = n\Phi_0$, where Φ_0 is the flux quantum and n is the vortex density) in the form

$$f_{VL} = \begin{cases} \dfrac{BH_{c1}}{4\pi} + Ku_s\varepsilon + \dfrac{K\varepsilon^2}{2}, & H \approx H_{c1} \\ \dfrac{1}{8\pi}\left[B^2 + BH_{c1}\dfrac{\ln(\nu d/\xi)^2}{\ln\kappa}\right] + Ku_s\varepsilon + \dfrac{K\varepsilon^2}{2}, & \\ H_{c1} \ll H \ll H_{c2}, & \end{cases} \quad (36)$$

where H_{c1} is the lower critical field. For a triangular lattice (considered for the sake of definiteness), we have $\beta_A = \langle\Psi^4\rangle_V/\langle\Psi^2\rangle_V = 1.16$ and $2\ln\nu = 2(\gamma - 1) + \ln[3^{1/2}/(8\pi)]$, where $\gamma = 0.577\ldots$ (Euler constant) [12]. The magnetic induction is $B = 2\Phi_0/(3^{1/2}d^2)$, where d is the distance between the vortices. It should be indicated that, in the first of relationships (36), we ignored the contribution due to vortex interactions at short distances.

In our case, H_{c1} in relationships (36) depends on u:

$$\begin{aligned} H_{c1}(u) &= \frac{\ln\kappa}{2^{1/2}\kappa}2a(u)\sqrt{\pi/b} \\ &= \frac{\ln\kappa}{2^{1/2}\kappa}(2a\sqrt{\pi/b^*} + 2r\sqrt{\pi/b}\varepsilon). \end{aligned} \quad (37)$$

Moreover, d/ξ also depends on u; i.e., $(d/\xi)^2 = d^2 4m|\Psi_s|^2 b\hbar^{-2}(1 + r\varepsilon/a)$. Minimizing expression (36) with respect to ε, we obtain

$$f_{VL} = \begin{cases} \dfrac{1}{8\pi}\left(2BH^0_{c1} - \dfrac{\ln^2\kappa\Delta K}{2\kappa^2 K}B^2\right), & H \approx H_{c1} \\ \dfrac{1}{8\pi}\left[B^2 + BH^0_{c1}\dfrac{\ln(\nu d/\xi)^2}{\ln\kappa} - \dfrac{\ln^2(\nu d/\xi)^2\Delta K}{4\kappa^2 K}B^2\right], & \\ H_{c1} \ll H \ll H_{c2}. & \end{cases} \quad (38)$$

Here $\Delta K/K$ is a relative jump of the bulk modulus in the transition from the normal to a superconducting phase (usually $\Delta K/K \ll 1$). The terms proportional to B^2 in Eqs. (38) correspond to the contributions of the noncore regions to the elastic vortex attraction.

When calculating the contributions due to vortex cores, the latter are usually considered as normal-phase cylinders with radii ξ [2, 3]. However, the corresponding contribution and the ratio of this contribution to the noncore contribution may be determined more exactly. With this aim one has to determine the strength of a dilatation source created by an isolated vortex. This strength is determined by the change in the volume due to vortex formation. The latter quantity is equal to the pressure derivative of the vortex energy $(\Phi_0/4\pi\lambda)^2(\ln\kappa + 0.08)$ [11], where Φ_0 is a flux quantum; λ depends on pressure; and the terms containing $\ln\kappa$ correspond to the noncore region, whereas the remaining terms correspond to the core region. Then, in the vicinity of H_{c1}, the ratio of the core to the noncore contributions to the vortex interactions equals $(0.08/\ln\kappa)^2$. If $H_{c1} \ll H \ll H_{c2}$ (the distance between vortices becomes less than the penetration depth ($1 \ll d/\xi \ll \kappa$)), the contribution of noncore regions decreases because of their overlap (see Eq. (38)), although it remains to be much larger than the core contribution.

Consider the vortex interactions in a finite medium with elastic anisotropy. As usual [11], we first simplify the free-energy expression. Integrating the terms with $\nabla\Psi$ in Eq. (29) by parts with the boundary condition $\mathbf{n}\left(-i\hbar\nabla - \dfrac{2e}{c}\mathbf{A}\right)\Psi|_\Sigma = 0$ (where $\mathbf{n}$ is the normal to the surface Σ) and using Eq. (30), we obtain

$$f = \frac{1}{V}\int\left[\frac{H^2}{8\pi} - \frac{b}{2}|\Psi|^4 + \frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl}\right]dv. \quad (39)$$

Equation (39) is the extension of the well-known Abrikosov equation for free energy [11] to a deformable medium.

At $H \ll H_{c2}$, it is convenient to single out the vortex contribution by writing $|\Psi|^2 = |\Psi_s|^2 - h$ and $u_{ij} = u^s_{ij} + u^v_{ij}$, where the quantities $|\Psi_s|^2$ and u^s_{ij} are determined by Eqs. (34) and (35), respectively. For vortices parallel

to the z axis, Eqs. (32) and (33) for u_{ij} yield

$$u_{ij}^{v} = r_{kl}\lambda_{ijkl}^{-1}\langle h\rangle + \frac{1}{2}\sum_{q\neq 0}[q_i S_k(\mathbf{q})G_{kj}(\mathbf{q}) + q_j S_k(\mathbf{q})G_{ki}(\mathbf{q})]h(\mathbf{q})e^{i\mathbf{qr}}, \quad (40)$$

where $S_k(\mathbf{q}) = r_{ij}q_j$, $G_{ki}(\mathbf{q}) = \lambda_{ijkl}q_j q_l$, and $\mathbf{q} = (q_x, q_y, 0)$ is a two-dimensional wave vector.

Taking into account the smallness of the coefficients r_{ij} in the calculations of h, one may limit oneself to the first approximation with respect to r_{ij}. Then, $h \cong h_0 + h_1$, where h_0 is the solution of Eq. (30) at $r_{ij} = 0$, and $h_1 = r_{ij}u_{ij}^{v}/b$ is the first correction to this solution, which, in the approximation under consideration, arises only in the noncore regions. Equation (40) also yields $r_{ij}u_{ij}^{v}(q) = b'(q)h(q) \cong b'(q)h_0(q)$, where

$$b'(\mathbf{q}) = \begin{cases} r_{ij}r_{kl}\lambda_{ijkl}^{-1}, & (\mathbf{q} = 0) \\ S_i(\mathbf{q})S_j(\mathbf{q})G_{ij}(\mathbf{q}), & (\mathbf{q} \neq 0). \end{cases} \quad (41)$$

The function $b'(q)$ at $\mathbf{q} \neq 0$ depends only on the orientation of the vector $\mathbf{q}$. As a result, one the following obtains [5] for the free energy described by Eq. (29):

$$f \cong f_s + b^*|\Psi_s|^2\zeta\langle h_0\rangle + \left\langle\frac{H^2}{8\pi}\right\rangle - \frac{b}{2}\langle h_0^2\rangle - \frac{1}{2}\sum_{\mathbf{q}} b'(\mathbf{q})h_0(\mathbf{q})h_0(-\mathbf{q}), \quad (42)$$

where $f_s = -b^*|\Psi_s|^4/2$ and $\zeta = 1 + b'(0)/b$. The latter term corresponds to the contribution of elastic strains to the energy of the vortex lattice, and the term containing $q = 0$ corresponds to the contribution of homogeneous strains. Since the elastic constants enter this term in the invariant combination (see Eq. (41) for $b'(0)$), this term is independent of the vortex orientation with respect to the crystal lattice. Such a dependence may arise only due to the terms with $\mathbf{q} \neq 0$.

In the vicinity of H_{c1}, the expression for the energy of vortex interactions may be simplified if one takes into account that these vortices form a regular lattice. With this aim, we represent $h_0(\mathbf{c})$ as a sum over the coordinates of the vortex centers, $h_0(\mathbf{c}) = \sum_i h_{01}(\mathbf{c} - \mathbf{c}_i)$. Then, $h_0(\mathbf{q}) = S^{-1}\sum_i \int h_{01}(\mathbf{c} - \mathbf{c}_i)e^{-i\mathbf{q}\rho} = S^{-1}\sum_i h_{01}(\mathbf{q})e^{-iq\rho_i}$ (where S is the area of the sample section in the x, y plane). Substituting this expression into Eq. (42), we may single out the interaction energy by subtracting the terms containing the factors $e^{-iq\rho_i}e^{-iq\rho_j}$ at $i = j$. Then, taking into account that $S^{-1}\sum_i \exp(-i\mathbf{q}\rho_i) = n\delta_{q,Q}$ (where n is the vortex density and $\mathbf{Q}$ are the reciprocal-lattice vectors), we obtain $h_0(\mathbf{q}) = nh_{01}(\mathbf{q})\delta_{q,\mathbf{Q}}$. The direct calculation shows that, with an increase in q, the noncore contribution to $h_{01}(q)$ dramatically decreases only if $q \sim \xi^{-1}$. Therefore $h_{01}(Q)$ may be approximated by its value at small wave vectors. As a result, we obtain for the energy of vortex interaction

$$f^{\text{int}} \cong -\frac{n^2}{2}h_{01}^2(0)\left[b'(0) + \sum_{Q\neq 0} b'(\mathbf{Q})\right] + \frac{n}{2S}\sum_{q\neq 0} b'(\mathbf{q}) \cong -\frac{n^2}{2}h_{01}^2(0)\left([b'(0) - \langle b'(\mathbf{Q})\rangle_\varphi] + \sum_{\mathbf{Q}\neq 0}[b'(\mathbf{Q}) - \langle b'(\mathbf{Q})\rangle_\varphi]\right), \quad (43)$$

where $\langle b'(\mathbf{Q})\rangle_\varphi = (2\pi)^{-1}\int_0^{2\pi} b'(\mathbf{Q})d\varphi$ is the $\mathbf{Q}$ value averaged over the orientations (it should be remembered that $b'(\mathbf{Q})$ is independent of the modulus of the vector $\mathbf{Q}$). We also used here the relationship $S^{-1}\sum_{q\neq 0} b'(\mathbf{q}) \approx \int b'(\mathbf{q})d\mathbf{q} = \int_0^{Q_{\max}} 2\pi q\langle b'(\mathbf{q})\rangle_\varphi dq \cong n\sum_Q \langle b'(\mathbf{Q})\rangle_\varphi$. An important conclusion following from the expressions (43) is that the vortex energy (bulk part) is independent of the sample shape. This follows from the fact that the spectrum of the wave vectors in Eq. (43) has no terms with small wave vectors of the order of the reciprocal of the sample size. The opposite conclusion about the dependence of the interaction on a sample shape drawn in [3] was based on the analysis of only a part of the elastic interactions, i.e., of the sum of pair interactions, each of which was calculated for an infinite medium.

The first term in parentheses in Eq. (43) corresponds to the interactions associated with the finite sample dimensions, i.e., with the action of imaging forces, whereas the second term has a nonzero value only in the presence of elastic anisotropy. In the isotropic case $b'(0) = r^2/K$ and $b'(\mathbf{Q} \neq 0) = r^2/(K + 4\mu/3)$. Since in the limit $\mu = \infty$ Eq. (43) should coincide with the first of Eqs. (38), we have

$$h_{01}(0) = 2\pi|\Psi_s|^2\xi^2\ln\kappa, \quad (44)$$

where it was taken into account that $nH_c\xi^2 = B/(2^{3/2}\pi\kappa)$ and $H_c = 2^{1/2}\kappa H_{c1}/\ln\kappa$. Equations (43) and (44) allow one to calculate the energy of the vortex interactions in a crystal of finite dimensions. However, since the function $b'(\mathbf{Q})$ depends on vector orientations in a rather complicated way, the final result may be obtained only numerically and only for crystals of certain symmetries. Evaluating each of two terms in parentheses on the right-hand side of Eq. (43) as $b\Delta K/K$ and taking into account Eq. (44), we obtain the vortex interaction in the

vicinity of H_{c1} as $f^{\text{int}} \approx -10^{-2}((\ln\kappa)/\kappa)^2(\Delta K/K)B^2$, i.e., the value exceeding the core contribution by a factor of $\sim 10^2 \ln^2(\kappa)$.

If $H_{c1} \ll H \ll H_{c2}$, the computations are more complicated because one also has to take into account the dependence of h_{01} on $\mathbf{q}$ in Eq. (42) with $h_0(\mathbf{q}) = S^{-1}\sum_i h_{01}(\mathbf{q})e^{-i\mathbf{q}\boldsymbol{\rho}_i}$. A similar calculation shows that in this case $h_{01}(0) = 2^{1/2}\pi|\Psi_s|^2\xi^2\ln\kappa$. However, now $Q > 1/\lambda$ and $h_{01}(Q) \approx h_{01}(0)\ln(\xi^{-1}Q^{-1})$. By using the same method, it is possible to obtain from Eq. (42), the relationship for calculating the energy of the vortex interaction for fields $H_{c1} \ll H \ll H_{c2}$ having a more general form than Eq. (43). At the same time, the result obtained for the isotropic case allows us to conclude that, in this field as well, the main contribution to the strain-induced vortex interaction in superconductors with high κ values comes from the change of the order parameter in noncore regions.

Thus, the results obtained show that in fields $H \ll H_{c2}$ the main contribution to the strain-induced vortex interaction in superconductors with high κ values comes from the change in the order parameter in noncore regions. The noncore contribution may exceed the core contribution by one or even two orders of magnitude. This conclusion is very important for studying the orientations of the vortex structures in crystals. For example, in the cases where the previous estimates showed that the difference between the elastic energies at various orientations of the vortex lattice is less than the differences of the corresponding London energies, the refinement of the elastic-interaction value may change the conclusion about the prevalence of one or another orientation.

Concluding the article, we would like to indicate that the method considered above, which is based on analysis of the case of an isotropic medium with an infinite shear modulus, considerably simplifies the calculation of the contribution of long-range elastic interaction to the energy of any one-dimensionally periodic structure in a finite medium with arbitrary anisotropy. Moreover, this method is also effective in the calculations of the striction contributions to the energy of two- and three-dimensional regular structures in an isotropic medium, e.g., for branching domains or systems of quantum dots. This statement is based on the facts that, as in the one-dimensional case, one may ignore here the near-surface distortions of such structures and that the elastic contribution to the free energy in an isotropic medium is independent of the orientation of the wave vectors of the structure with respect to the crystal lattice.

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